LOTFIZADEH OLYMPIAD
BAKU / AZERBAIJAN
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Problems and Solutions

Problem Selection committee: Morteza Saghafian,
Mona Mohammadi, Abbas Mammadov, Anar Rzayev

Problem 1. In the inscribed quadrilateral $ABCD$, P is the intersection point of diagonals and M is the midpoint of arc AB. Prove that line MP passes through the midpoint of segment CD, if and only if lines AB,CD are parallel.

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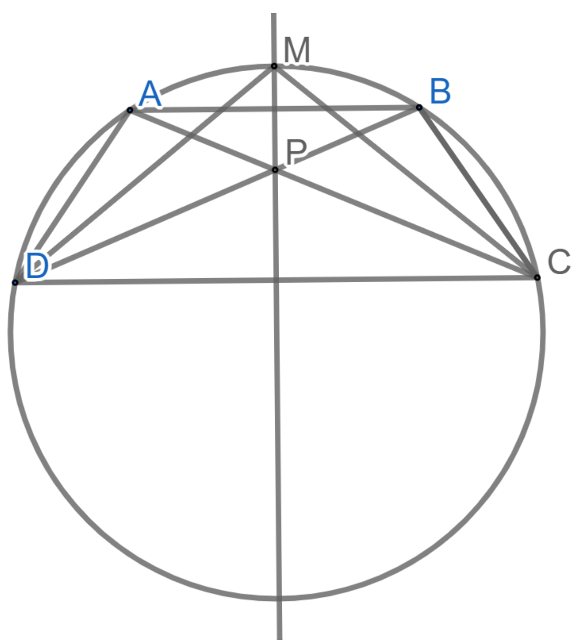
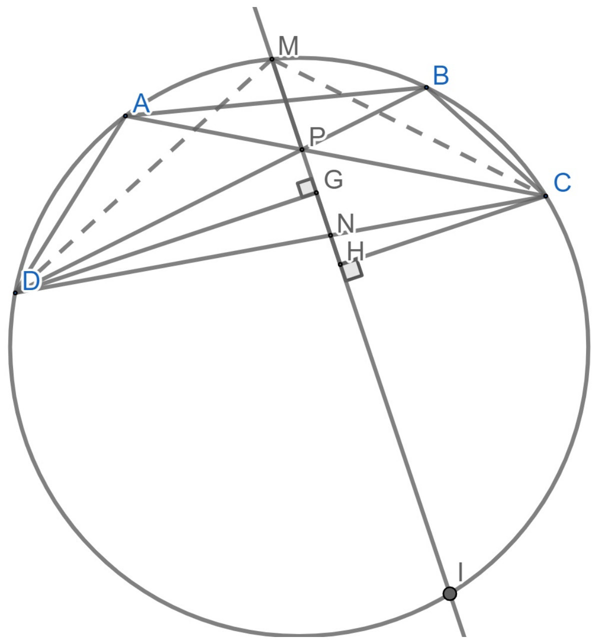
Solution.

Let N be the midpoint of CD. First assume that AB and CD are parallel.



Then $ABCD$ is a trapezoid and M, P, N all lie on the perpendicular bisector of CD.
Now Assume that M, P, N are collinear.

First solution:



Draw the perpendiculars DG and CH to line MN. Since N is the midpoint of CD we have $DG = CH$. Therefore, the area of $\triangle MPD$ and $\triangle MPC$ are equal. Now without loss of generality assume that $\angle DPN > \angle NPC$. Since $\angle DPN + \angle NPC < 180$, we have $CP < DP$.

$$\angle DMP = \angle DPN - \angle MDP \tag{1}$$

$$\angle CMP = \angle CPN - \angle MCP \tag{2}$$

$$\angle DPN > \angle NPC \tag{3}$$

$$\overset{\frown}{AM} = \overset{\frown}{MB} \Longrightarrow \angle MDP = \angle MCP \tag{4}$$

$$\tag{5}$$

Hence, $\angle CMN < \angle DMN$ and similarly we have $MC < MD$. By multiplying the two inequalities we get $MC.CP < MD.DP$, and by multiplying both sides by $sin\angle MCP$ we obtain that the area of $\triangle MPD$ is greater than the area of $\triangle MPC$ which is a contradiction. So $\angle NPC$ and $\angle DPN$ should be equal and therefore, MN is perpendicular bisector of CD. Hence we conclude that AB and CD are parallel.

Second solution: Using sine law we get:

$$\triangle PCD \Longrightarrow \frac{sin\angle PDC}{sin\angle PCD} = \frac{PC}{PD} \tag{6}$$

$$\triangle PMC, \triangle PMD \Longrightarrow \frac{PC}{PD} = \frac{sin\angle PMC}{sin\angle PMD} \tag{7}$$

$$\triangle MCN, \triangle MND \Longrightarrow \frac{sin\angle NMC}{sin\angle NMD} = \frac{sin\angle MCN}{sin\angle MDN} \tag{8}$$

Let $\angle MCP = \alpha$, $\angle PDC = x$,$\angle PCD = y$. Since M, P, N are collinear, we obtain:

$$sin(x + \alpha).sin(x) = sin(y + \alpha).sin(y) \tag{9}$$

$$\frac{1}{2}(cos(2x + \alpha) - cos(\alpha)) = \frac{1}{2}(cos(2y + \alpha) - cos(\alpha)) \tag{10}$$

This gives us $x = y$ or $x + y + \alpha = 180$, but the second case is not possible since in triangle $\triangle ACD$ the sum of angles is 180 degrees.

Problem 2. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be (not necessarily distinct) positive integers. We continue the sequences as follows: For every $i > n$, a_i is the smallest positive integer which is not among $b_1, b_2, \cdots, b_{i-1}$, and b_i is the smallest positive integer which is not among $a_1, a_2, \cdots, a_{i-1}$. Prove that there exists N such that for every $i > N$ we have $a_i = b_i$ or for every $i > N$ we have $a_{i+1} = a_i$.

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Solution.

Assume that for some $j > n$, $a_j \neq b_j$. Then we have $a_{j+1} = a_j$ because a_j is still the smallest positive integer which is not among $b_1, b_2, \cdots, b_j$, with similar reason we have $b_{j+1} = b_j$. Hence $a_{j+1} = a_j \neq b_j = b_{j+1}$ and we can argue that for $i > j$, $a_i = a_j \neq b_j = b_i$.

Problem 3. Find the least possible value for the fraction

$$\frac{lcm(a,b)+lcm(b,c)+lcm(c,a)}{gcd(a,b)+gcd(b,c)+gcd(c,a)}$$

over all distinct positive integers a, b, c.

By $lcm(x, y)$ we mean the least common multiple of x, y and by $gcd(x, y)$ we mean the greatest common divisor of x, y.

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Solution.

The answer is $\frac{5}{2}$. Indeed $(a, 2a, 4a)$ gives us $\frac{5}{2}$, we want to prove that this is the minimum value. We can assume that $gcd(a, b, c) = 1$. If all fractions $\frac{lcm(a,b)}{gcd(a,b)}$ are at least 3, then the value is also at least 3. So suppose that one of them is 2. Wlog suppose that $\frac{lcm(a,b)}{gcd(a,b)} = 2$. This means wlog $b = 2a$. So $gcd(a, c) = 1$ and $lcm(a, c) = ac$. Here we have two cases:

Case (1): $gcd(2a, c) = 1$. So $lcm(2a, c) = 2ac$ and the fraction equals $\frac{2a+ac+2ac}{a+2}$. The only case where the above fraction is less than $\frac{5}{2}$ is when a=c=1, which is a contradiction (numbers are distinct).

Case (2): $gcd(2a, c) = 2$. So $lcm(2a, c) = ac$, so c is even (at least 2) and the fraction equals $\frac{2a+ac+ac}{a+3}$. The only cases where the above fraction is less than $\frac{5}{2}$ are $(a, c) = (1, 2)$ or $(2, 2)$, both contradict the distinctness ($b = 2a$).
(For $(a, c) = (1, 4)$ the above fraction equals $\frac{5}{2}$, which yields to $(1, 2, 4)$.)

Problem 4. Find the number of sequences of $0, 1$ with length n satisfying both of the following properties:

- There exists a simple polygon such that its i-th angle is less than 180 degrees if and only if the i-th element of the sequence is 1.
- There exists a convex polygon such that its i-th angle is less than 90 degrees if and only if the i-th element of the sequence is 1.

Solution.

The answer is $\binom{n}{3}$. First we claim that every simple polygon has at least three angles less than 180 degrees. We prove this claim by taking the sum of angles for a simple n-gon which is $(n-2) \cdot 180$, so there cannot be at least $n-2$ angles greater than 180 in it, therefore at least three of them have value less that 180.

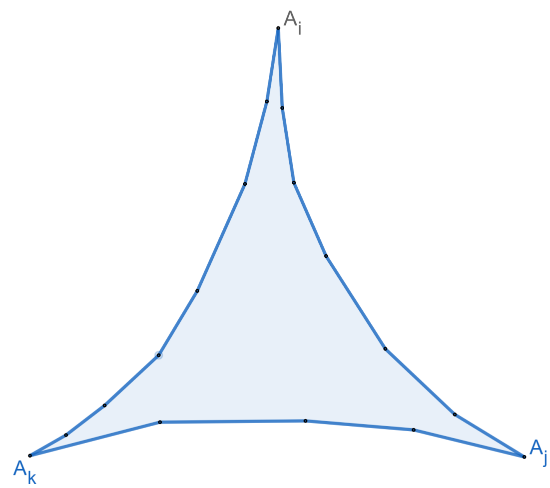
Second we claim that every convex polygon has at most three angles less than 90 degrees. Again we prove this claim by taking the sum of angles. Suppose that a convex n-gon has x angles less than 90 degrees and $n-x$ angles between 90 and 180 degrees. The sum of its angles would be

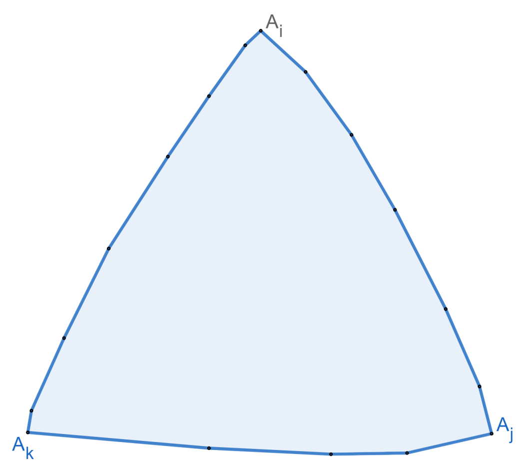
$$(n-2) \cdot 180 < x \cdot 90 + (n-x) \cdot 180.$$

This gives us $x \cdot 90 < 360$ which leads to $x \leq 3$.

By the above claims we conclude that every sequence satisfying both conditions has exactly three 1s.

Finally, we argue that every sequence with three 1s and $n-3$ zeros satisfy both conditions. Suppose we have a sequence of length n with elements $i < j < k$ to be 1 and the other elements to be 0. Below figures show two polygons. The first one is a simple polygon $A_1A_2 \cdots A_n$ whose angles A_i, A_j, A_k are less than 180 and the other angles are more than 180. The second one is a convex polygon $A_1A_2 \cdots A_n$ whose angles A_i, A_j, A_k are less than 90 and the other angles are more than 90. Note that in both figures the triangles $A_iA_jA_k$ are equilateral triangles and the paths between them are close enough to the segment between them.





Therefore, there are $\binom{n}{3}$ such sequences as claimed.