Lotfi Zadeh Olympiad

Mathematics Contest

December 22, 2021

The problems of this contest are to be kept confidential until they are posted on the official Lotfi Zadeh Olympiad website: lotfizadeh.org

Problem 1. In the inscribed quadrilateral $ABCD$, P is the intersection point of diagonals and M is the midpoint of arc AB. Prove that line MP passes through the midpoint of segment CD, if and only if lines AB,CD are parallel.

Problem 2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be (not necessarily distinct) positive integers. We continue the sequences as follows: For every $i > n$, a_i is the smallest positive integer which is not among $b_1, b_2, ..., b_{i-1}$, and b_i is the smallest positive integer which is not among $a_1, a_2, ..., a_{i-1}$. Prove that there exists N such that for every $i > N$ we have $a_i = b_i$ or for every $i > N$ we have $a_{i+1} = a_i$.

Problem 3. Find the least possible value for the fraction

$$\frac{lcm(a,b) + lcm(b,c) + lcm(c,a)}{gcd(a,b) + gcd(b,c) + gcd(c,a)}$$

over all distinct positive integers a, b, c.
By $lcm(x, y)$ we mean the least common multiple of x, y and by $gcd(x, y)$ we mean the greatest common divisor of x, y.

Problem 4. Find the number of sequences of $0, 1$ with length n satisfying both of the following properties:

- There exists a simple polygon such that its i-th angle is less than 180 degrees if and only if the i-th element of the sequence is 1.
- There exists a convex polygon such that its i-th angle is less than 90 degrees if and only if the i-th element of the sequence is 1.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.